Study Group Talk: Flatness

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Today I'm going to speak about what it means to be flat, in particular how this adjective transfers from the world algebra to the world of geometry. A motivating picture is the following: the first example looks like the 'nicer' of the two and is in fact flat – we'll discuss how flatness can be detected by looking at the fibres.

1 Algebra

Let R be a commutative rings with unity and let M be an R-module.

Most traditionally, flatness is defined as follows:

Definition 1.1. M is called flat over R if whenever the sequence of R-modules

$$0 \to A \to B \to C \to 0$$

is short exact, then the sequence of R-modules obtained by right tensoring with M

$$0 \to A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$$

is also short exact.

More precisely, as this part is automatically exact, the definition imposes that for M to be flat, tensoring by it must preserve injections.

Given another (commutative) ring S (with unity) and a map $f : R \to S$, we can endow S with the structure of an R-module, i.e. equipping it with the operation

$$R \times S \to S, \qquad (r, s) \mapsto f(r)s.$$

This idea allows us to further define flatness of a ring map:

Definition 1.2. A map of rings $f : R \to S$ is called flat if S is flat over R.

Flatness is a particularly nice condition when the ring R is a PID.

Recall,

Definition 1.3. We say $m \in M$ is a torsion element if $rm = 0$ for some non zero-divisor $r \in R$. Moreover, we say that M is torsion-free if the only torsion element is 0.

Proposition 1.4. *If R is a PID then M is flat if and only if it is torsion free.*

Sketch proof of 'only if': Let M be a flat R module and pick $0 \neq a \in R$. The map

$$R \to R, \qquad r \mapsto ar$$

is an injection, so the induced map

$$R \otimes_R M \to R \otimes_R M$$

is too. The latter map is in fact

$$M \to M, \qquad m \mapsto am$$

and this being injective for every a is exactly the statement that M is torsion free.

Remark. *Notice, this direction of the proposition doesn't use the fact that R is a PID, but this is essential for the proof of the converse statement.*

This proposition gives a straight forward way to identify flat modules over fields or $\mathbb{Z}$ for example.

Example 1.5. $\mathbb{Q}$ is a flat $\mathbb{Z}$-module (abelian group), as for $0 \neq n \in \mathbb{Z}$, $n\frac{a}{b} = 0 \Leftrightarrow \frac{a}{b} = 0$.

Example 1.6. $\mathbb{Q}^*$ is not a flat $\mathbb{Z}$-module (abelian group) as $(-1)^2 = 1$, i.e. -1 is a nonzero torsion element.

Scenarios that are possibly of more interest to us include the following:

Example 1.7. Consider $\mathbb{C}[x, y]/(xy)$ as a $\mathbb{C}[x]$-module, i.e. equip it with the operation

$$(f(x), g(x, y) + (xy)) \mapsto f(x)g(x, y) + (xy).$$

This module is not torsion-free, $y + (xy)$ is a nonzero torsion element, and therefore not flat.

Example 1.8. As above, consider $\mathbb{C}[x, y, t]/(y^2 - x^3 + t)$ as a $\mathbb{C}[t]$-module.

Suppose that $g(x, y, t) + (y^2 - x^3 + t) \in \mathbb{C}[x, y, t]/(y^2 - x^3 + t)$ is a torsion element, i.e. there's some $f(t) \in \mathbb{C}[t]$ such that $f(t)g(x, y, t) \in (y^2 - x^3 + t)$. Assuming that $f(t)$ is non-constant (if it were, we automatically obtain that $g(x, y, t)$ is zero), we can factor it and write

$$(y^2 - x^3 + t)q(x, y, t) = \prod_{i=1}^{n} (\lambda_i t - \mu_i) g(x, y, t).$$

Clearly, $y^2 - x^3 + t$ is not divisible by $\lambda_i t - \mu_i$ for any index i, so $f(t)$ divides $q(x, y, t)$ and $g(x, y, t) + (y^2 - x^3 + t)$ is zero.

This module is therefore torsion-free and flat.

Example 1.9. Similarly, consider $\mathbb{C}[x, y, t]/(t^2y^2 - t^2x^3 + t)$ as a $\mathbb{C}[t]$-module. This is not flat, $ty^2 - tx^3 + 1 + (t^2y^2 - t^2x^3 + t)$ is a nonzero torsion element.

Moreover,

Lemma 1.10. *Any $\mathbb{C}[t]$-module of the form $\mathbb{C}[x_1, \ldots, x_n, t]/(a(x_1, \ldots, x_n, t))$ is flat if and only if there's no non-constant $b(t) \in \mathbb{C}[t]$ dividing $a(x_1, \ldots, x_n, t)$.*

The proof of this is just an adaptation of our arguments above.

Returning to R in general, we still have different formulations of flatness.

Proposition 1.11 (Flatness is a local property)**.** *M is flat over R if and only if for each prime ideal $\mathfrak{p} \subseteq R$, $M_\mathfrak{p}$ is flat over $R_\mathfrak{p}$.*

Remark. *Local property comes from this being characterised by the localizations $M_\mathfrak{p}$, i.e.*

$$\left\{ \frac{x}{y} \,|\, x \in M, y \in R \backslash \mathfrak{p} \right\} / \sim,$$

where $\frac{x_1}{y_1} \sim \frac{x_2}{y_2}$ if and only if there's some $u \in R\backslash\mathfrak{p}$ such that $u(x_1y_2 - x_2y_1) = 0$.

2 Geometry

Recall, affine schemes look like the spectrum of some ring R, i.e a topological space with underlying set

$$\operatorname{Spec} R = \{\text{prime ideals } \mathfrak{p} \subseteq R\}$$

(whose closed sets are given by $V(I) = \{\mathfrak{p} \in \text{Spec } R : I \subseteq \mathfrak{p}\}$ for each ideal $I \subseteq R$) equipped with a 'structure sheaf' $\mathcal{O}_{\operatorname{Spec} R}$.

A morphism between the affine schemes $\operatorname{Spec} R$ and $\operatorname{Spec} S$ is a pair $(f, f^\#)$ where

$$f : \operatorname{Spec} R \to \operatorname{Spec} S$$

is a continuous map of topological spaces and

$$f^\# : \mathcal{O}_{\operatorname{Spec} S} \to \mathcal{O}_{\operatorname{Spec} R}$$

is a sheaf morphism such that every induced map of stalks (= localizations for affine schemes), i.e. for each $\mathfrak{p} \in \operatorname{Spec} R$, $f^\#_\mathfrak{p} : S_{f(\mathfrak{p})} \to R_\mathfrak{p}$ is a local ring homomorphism.

Definition 2.1. A morphism $(f, f^\#)$ of the affine schemes $\operatorname{Spec} R$ and $\operatorname{Spec} S$ is called flat if $f^\#_\mathfrak{p}$ is a flat ring map for each $\mathfrak{p} \in \operatorname{Spec} R$.

This definition isn't very enlightening. Luckily, Proposition 1.11 allows us to show that the two notions we have of flatness are consistent and we'll instead take this as our definition.

Proposition 2.2. *The map of rings $R \to S$ is flat, i.e. S is flat over R, if and only if the corresponding map of affine schemes $\operatorname{Spec} S \to \operatorname{Spec} R$ is flat.*

Recycling some of our earlier examples:

Example 2.3. The map of schemes $\operatorname{Spec}\mathbb{C}[x,y]/(xy) \to \operatorname{Spec}\mathbb{C}[x]$ is not flat.

Example 2.4. The map of schemes $\operatorname{Spec}\mathbb{C}[x,y,t]/(y^2 - x^3 + t) \to \operatorname{Spec}\mathbb{C}[t]$ is flat.

Example 2.5. The map of schemes $\operatorname{Spec}\mathbb{C}[x,y,t]/(t^2y^2 - t^2x^3 + t) \to \operatorname{Spec}\mathbb{C}[t]$ is not flat.

Identifying $\operatorname{Spec}\mathbb{C}[x_1,\ldots,x_n,t]/(a(x_1,\ldots,x_n,t))$ with the variety $a(x_1,\ldots,x_n,t)=0$ and $\operatorname{Spec}\mathbb{C}[t]$ with $\mathbb{C}$, these maps and their fibres are the following

$$\{xy=0\} \to \mathbb{C},\ (x,y) \mapsto y;$$

$$\{y^2 = x^3 - t\} \to \mathbb{C},\ (x,y,t) \mapsto t;$$

$$\{t^2y^2 = t^2x^3 - t\} \to \mathbb{C},\ (x,y,t) \mapsto t;$$

Recognise the second and third images here as those which I referred to at the beginning of this talk. We distinguish these from the first, as their sources are defined by smooth varieties (check the rank of the Jacobian matrix).

Observations we can make are that there are some consistencies amongst the fibres. In the second case, each fibre is an elliptic curve, nonsingular for $t \neq 0$ and most notably, each has dimension 1. In the third case, all fibres are elliptic curves with $t = 0$ being an exception, it is an entire plane, so all fibres have dimension 1 or 2.

This is no coincidence, 'Miracle Flatness' allows us to detect flatness from the dimension of the fibres.

Theorem 2.6 (Miracle flatness)**.** *Let $X \to Y$ be a morphism between smooth, irreducible (i.e. connected) varieties. This morphism is flat if and only if each fibre, if it exists, has dimension* $\dim X - \dim Y$*.*

Here we take a variety to be the zero locus of some ideal $I \subseteq k[x_1, x_2, \cdots, x_n]$, for k a field.

Remark. *We define* $\dim X$ *to be the maximal length, d, of the chains*

$$X_0 \subset X_1 \subset \ldots \subset X_d$$

of distinct nonempty (irreducible) subvarieties of X.

In fact the smoothness assumption can be dropped in the 'only if' direction.

Example 2.7. By our earlier lemma, $\mathbb{C}[x, y, t]/(y^2 - tx^3)$ is a flat $\mathbb{C}[t]$-module and so is the corresponding map

$$\{y^2 = tx^3\} \to \mathbb{C}, \ (x, y, t) \mapsto t.$$

Again we see that the fibres all have dimension 1, but the defining variety is singular along $x = y = 0$.

Smoothness is however essential in the 'if' direction.

Example 2.8. Consider the map $\operatorname{Spec} \mathbb{C}[t] \to \operatorname{Spec} \mathbb{C}[x, y]/(y^2 - x^3)$ defined by

$$\mathbb{C} \to \{y^2 = x^3\}, \quad t \mapsto (t^2, t^3).$$

First, notice that the curve $y^2 = x^3$ is singular at $(0, 0)$.

Second, notice that this map is a bijection. So every fibre is a single point, i.e has dimension zero.

Finally, I claim that this map is not flat. To show this, on the level of rings, we want that $\mathbb{C}[x,y]/(y^2 - x^3)$ is not flat over $\mathbb{C}[t]$. This is not easy to spot using the tools we've developed so far. Instead we use the following:

Proposition 2.9. *Let R be an integral domain and $\tilde{R}$ its integral closure. If $\tilde{R}$ is flat over R then R is integrally closed, i.e. $R = \tilde{R}$.*

In the notation of the proposition, take $R = \mathbb{C}[x,y]/(y^2 - x^3) \cong \mathbb{C}[t^2, t^3]$, so it's field of fractions is $K = \mathbb{C}(t)$. R is not integrally closed in K since $t \in K$ has minimal polynomial $x^2 - t^2 \in R[x]$, but $t \notin R$.

In fact, the integral closure of R is $\tilde{R} = \mathbb{C}[t]$. We can therefore apply the contrapositive of the theorem to obtain that $\mathbb{C}[x,y]/(y^2 - x^3)$ is not flat over $\mathbb{C}[t]$.

Miracle flatness motivates why flat maps of smooth varieties are particularly 'nice'. Not only do the fibres of a flat map have the same dimension, other properties are preserved too: for example, degree (you can't have one fibre a conic and another a cubic) and arithmetic genus.