Parity conjecture for hyperelliptic curves I

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Parities of ranks

Let C be a curve over a number field.

The Parity Conjecture (PC)

$$(-1)^{\mathrm{rk}(\mathrm{Jac}\,C)} = \prod_v w_v(\mathrm{Jac}\,C)$$

$$\Uparrow$$

Goal: Arithmetic analogue of PC

$$(-1)^{\mathrm{rk}(\mathrm{Jac}\,C)} = \prod_v \lambda_v(\mathrm{Jac}\,C)$$

$$+$$

Find an error term

$$w_v = H_v \cdot \lambda_v \quad \text{and} \quad \prod_v H_v = +1$$

Work in progress (Dokchitser, G., Morgan)

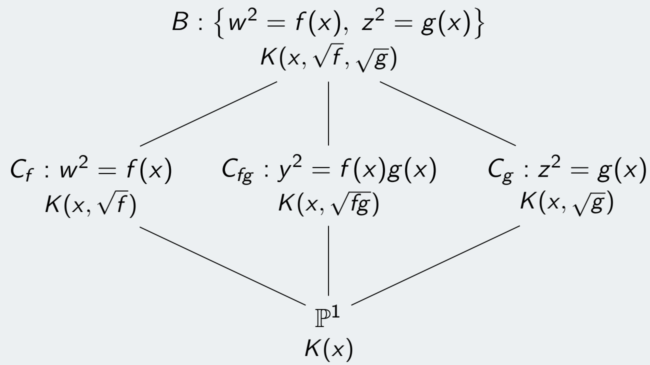
Assume $\#\text{Ш} < \infty$. The parity conjecture holds for all semistable hyperelliptic curves.*

Nice hyperelliptic curves

Set up:

- K a number field,
- $f(x), g(x) \in K[x]$ with $f(x)g(x)$ separable.

Assume $\#Ш < \infty$.

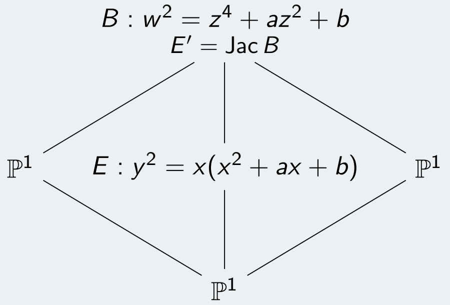


Theorem (Dokchitser, G., Morgan)

$$\mathrm{rk}(\mathrm{Jac}\, C_{fg}) \equiv \mathrm{rk}(\mathrm{Jac}\, C_f) + \mathrm{rk}(\mathrm{Jac}\, C_g) + \sum_v \lambda_v \mod 2$$

where λ_v is an explicit expression in the data attached to C_f, C_g, C_{fg}, B over K_v.

Example: Elliptic curves



Let $K = \mathbb{Q}$, $f(x) = x^2 + ax + b$, $g(x) = x$.

- There's a 2-isogeny $E \to E'$.
- (Cassels) Assuming $\#Ш < \infty$,

$$\mathrm{BSD}(E) = \mathrm{BSD}(E')$$

i.e.,

$$\square \cdot 2^{\mathrm{rk}(E)} = \frac{\mathrm{Reg}(E)}{\mathrm{Reg}(E')} = \frac{\#E(\mathbb{Q})^2_{\mathrm{tors}}}{\#E'(\mathbb{Q})^2_{\mathrm{tors}}} \cdot \frac{\#Ш(E')}{\#Ш(E)} \cdot \frac{\Omega(E')}{\Omega(E)} \cdot \frac{\prod_p c_p(E')}{\prod_p c_p(E)}$$

$$\Rightarrow \quad \mathrm{rk}(E) \equiv \mathrm{rk}(\mathrm{Jac}\,\mathbb{P}^1) + \mathrm{rk}(\mathrm{Jac}\,\mathbb{P}^1) + \lambda_\infty + \sum_p \lambda_p \mod 2$$

The Parity Conjecture

Work in progress (Dokchitser, G., Morgan)

Assume $\#\text{Ш} < \infty$. The parity conjecture holds for all semistable hyperelliptic curves.*

$$\mathrm{rk}(\mathrm{Jac}\, C_{fg}) \;+\; \mathrm{rk}(\mathrm{Jac}\, C_f) \;+\; \mathrm{rk}(\mathrm{Jac}\, C_g) \;\equiv\; \sum_v \lambda_v \mod 2 \qquad (*)$$

Strategy

Reduce to PC for $C : y^2 = h(x)$ with $\mathrm{Gal}(h)$ a 2-group $\implies C = C_{fg}$ or $C_{f\bar{f}}$

- Find error term for $(*) \implies$ PC holds for Jac C_{fg} iff it holds for Jac $C_f \times$ Jac C_g

  i.e., $w_v(\mathrm{Jac}\, C_{fg}) w_v(\mathrm{Jac}\, C_f) w_v(\mathrm{Jac}\, C_g) = H_v \cdot (-1)^{\lambda_v}$ and $\prod_v H_v = +1$

- Find error term for $\ldots \implies$ PC holds for Jac $C_{f\bar{f}}$ iff it holds for Jac $C_f / K(\sqrt{\alpha})$
- Argue by induction on $\deg(h)$

Thank you for your attention!