Birch and Swinnerton-Dyer for curves

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Conjecture (Birch and Swinnerton–Dyer, Tate)

Assuming that $L(\mathrm{Jac}\, X/\mathbb{Q}, s)$ has an analytic continuation to $\mathbb{C}$,

- $\mathrm{rank}(\mathrm{Jac}\, X/\mathbb{Q}) = \mathrm{ord}_{s=1} L(\mathrm{Jac}\, X/\mathbb{Q}, s)$,
- *the leading term in the Taylor expansion of $L(\mathrm{Jac}\, X/\mathbb{Q}, s)$ at $s = 1$ is*

$$\mathrm{BSD}(\mathrm{Jac}\, X/\mathbb{Q}) = \frac{\#\text{Ш}(\mathrm{Jac}\, X)\Omega(\mathrm{Jac}\, X)\mathrm{Reg}(\mathrm{Jac}\, X)\prod_p c_p(\mathrm{Jac}\, X)}{\#\mathrm{Jac}\, X(\mathbb{Q})^2_{\mathrm{tors}}}.$$

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The L-function has an expression as an Euler product

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Lemma

Let $\mathcal{X}$ be a regular model of X over $\mathbb{Z}_p$, if Frob_p acts trivially on $\mathcal{X}$

$$L_p(\operatorname{Jac} X, T) = (1-pT)^{N_I}(1-T)^{N_C} Z_p(\mathcal{X}, T),$$

where $N_I = \#\text{irreducible comps of } \mathcal{X}$, $N_C = \#\text{connected comps of } \mathcal{X}$.

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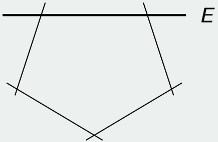

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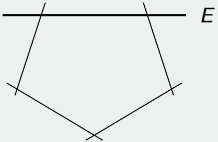

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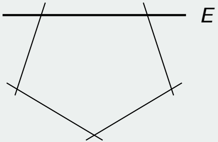

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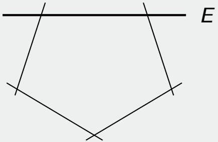

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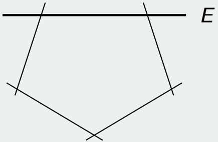

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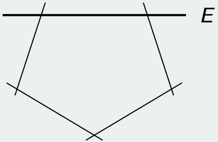

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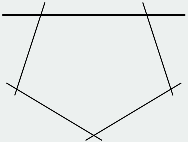

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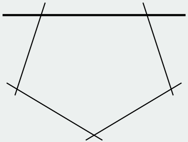

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So $L_5(\operatorname{Jac} X, T) = L_5(E, T)(1-T)$, i.e.

$$L_5(\operatorname{Jac} X, T) = (1+2T+5T^2)(1-T) = 1 + T + 3T^2 - 5T^3.$$

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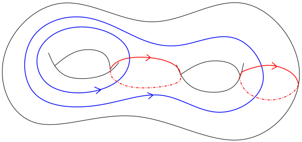

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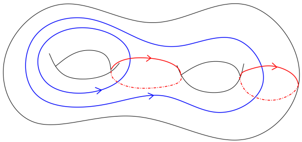

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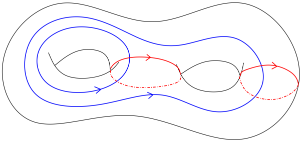

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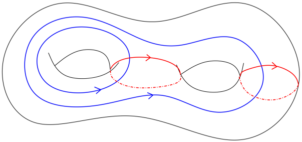

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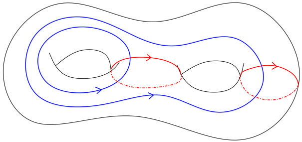

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Lemma

The lattice inside $\mathbb{R}$ spanned by the P_I is generated by $\operatorname{covol}(\Lambda_\omega \cap \mathbb{R}^g)$.

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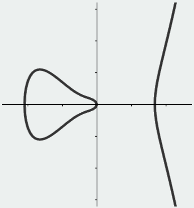

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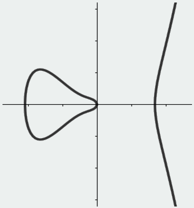

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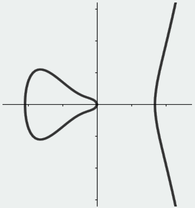
$$P_{\{1,2\}} = \left|\int_{\gamma_1} \frac{dx}{y} \int_{\gamma_2} x\frac{dx}{y} - \int_{\gamma_1} x\frac{dx}{y} \int_{\gamma_2} \frac{dx}{y}\right|$$



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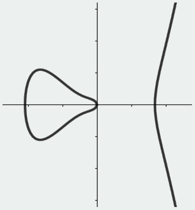

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So $\mathrm{covol}(\Lambda_\omega \cap \mathbb{R}^2) \approx 22.712$

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This recovers $\Omega(E/\mathbb{Q}) = \left|\int_{E^0(\mathbb{R})} \frac{dx}{2y}\right| \times \#\mathrm{Comp}\, E(\mathbb{R})$.

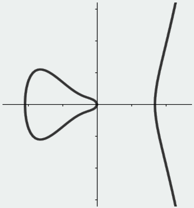
Example 2

Let $X : y^2 = x^5 + x^4 - 3x^3 - 2x^2 - x$.

Let $\omega_1, \omega_2 = \frac{dx}{y}, x\frac{dx}{y}$ and γ_1, γ_2 be the real loops.

$$P_{\{1,2\}} = \left|\int_{\gamma_1} \frac{dx}{y} \int_{\gamma_2} x\frac{dx}{y} - \int_{\gamma_1} x\frac{dx}{y} \int_{\gamma_2} \frac{dx}{y}\right|$$

So $\mathrm{covol}(\Lambda_\omega \cap \mathbb{R}^2) \approx 22.712 \Rightarrow \Omega(\mathrm{Jac}\, X) \approx 11.356$.



Tamagawa numbers

The Tamagawa product is

$$\prod_p c_p(\operatorname{Jac} X/\mathbb{Q})$$

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Lemma

$c_p(\operatorname{Jac} X/\mathbb{Q})$ *is the size of the* Frob_p *invariants of the cokernel of*

$$H_1(\Upsilon, \mathbb{Z}) \to \operatorname{Hom}(H_1(\Upsilon, \mathbb{Z}), \mathbb{Z}); \qquad \ell \mapsto \langle \ell, \cdot \rangle.$$

Tamagawa numbers

Example 1

Example 2

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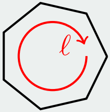

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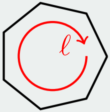

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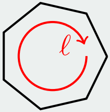

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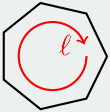

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Tamagawa numbers

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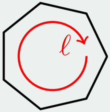

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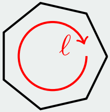

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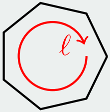

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Tamagawa numbers

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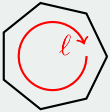

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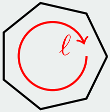

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Tamagawa numbers

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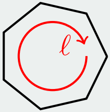

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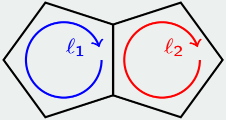

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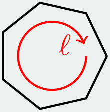

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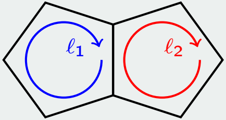

The image of $b_1\ell_1 + b_2\ell_2$ under $a_1\ell_1 + a_2\ell_2$ is

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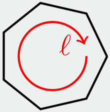

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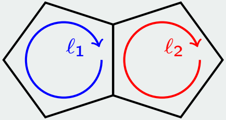

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Definition

A curve $X/\mathbb{Q}$ of genus g is *deficient* at v if it has no $\mathbb{Q}_v$-rational divisor of degree $g-1$.

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So $\#\text{Ш} = \square$ for elliptic curves and for odd genus hyperelliptic curves.

Thank you for listening!

Any questions?