

MODELS OF CURVES: THE BIRCH AND SWINNERTON-DYER FORMULA

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1. THE STATEMENT

Notation 1.1. • K a number field of discriminant Δ_K

- E/K an elliptic curve
- $|\cdot|_v$ the normalised absolute value on K_v (for v a place of K)
- q_v the cardinality of the residue field of K_v

Conjecture 1 (Birch and Swinnerton-Dyer part II). *Assuming III_E is finite and that $L(E, s)$ has an analytic continuation to \mathbb{C} , its lead coefficient at $s = 1$ is*

$$\frac{\#\text{III}_E \cdot \text{Reg}_E \cdot C_E}{\#E(K)_{\text{tors}}^2 \cdot \sqrt{|\Delta_K|}}.$$

Today's focus is the product of local terms C_E , which is model dependent.

Definition 1.2. Fix a non-zero differential ω on E . Then

$$C_E := \prod_{v|\infty} c_{E/K_v} \left| \frac{\omega}{\omega_v^0} \right|_v \cdot \prod_{\substack{v|\infty \\ K_v \cong \mathbb{R}}} \int_{E(K_v)} |\omega|_v \cdot \prod_{\substack{v|\infty \\ K_v \cong \mathbb{C}}} \int_{E(K_v)} |\omega \wedge \bar{\omega}| = \prod_v C_{E/K_v}(\omega)$$

where $c_{E/K_v} = [E(K_v) : E_0(K_v)]$ is a *Tamagawa number* and ω_v^0 is a *Néron differential*.

If $E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ is a minimal Weierstrass equation over K_v , then

$$\omega_v^0 = \pm \frac{dx}{2y + a_1x + a_3}.$$

Example 1.3. Let $E/\mathbb{Q} : y^2 = x^3 + 17^{12}$, $\Delta_E = -2^3 \cdot 3^3 \cdot 17^{24}$. This is minimal over \mathbb{Q}_p , and so $\omega_p^0 = \pm \frac{dx}{2y}$ whenever $p \neq 17$.

Let $y = 17^6Y$, $x = 17^4X$. Then $E : Y^2 = X^3 + 1$ is minimal over \mathbb{Q}_{17} and

$$\omega_{17}^0 = \pm \frac{dX}{2Y} = \pm 17^2 \frac{dx}{2y}.$$

Remark 1.4. When $K = \mathbb{Q}$, E has a global minimal Weierstrass equation (e.g. $E : y^2 = x^3 + 1$ in the example above). Therefore, there's a canonical choice of ω which gives

$$C_E = \prod_p c_{E/\mathbb{Q}_p} \cdot \int_{E(\mathbb{R})} |\omega|.$$

Remark 1.5. Each local term $C_{E/K_v}(\omega)$ depends on the choice of ω , but C_E doesn't. To see this: for $\alpha \in K^\times$, $C_{E/K_v}(\alpha\omega) = |\alpha|_v C_{E/K_v}(\omega)$ and $\prod_v |\alpha|_v = 1$.

Where does this term come from? (From my understanding).

Recall that $L(E, s) = \prod_{v \nmid \infty} L(E, q_v^{-s})^{-1}$. The non-Archimedean terms appearing in the Birch and Swinnerton-Dyer conjecture arise naturally as v -adic integrals.

Lemma 1.6 (Tate, [5] Theorem 5.2). *Given $v \nmid \infty$,*

$$L_v(E, q_v^{-1})^{-1} \int_{E(K_v)} |\omega^0|_v = c_{E/K_v} \left| \frac{\omega}{\omega_v^0} \right|_v.$$

In the 1950's, before the formulation of the Birch and Swinnerton-Dyer conjecture, the Tamagawa number of a semisimple algebraic group was receiving a lot of attention. It is possible that Birch and Swinnerton-Dyer tried to mimic this construction for elliptic curves as follows:

$$\prod_v \int_{E(K_v)} |\omega|_v \overset{\substack{\text{doesn't converge} \\ \rightsquigarrow \\ \text{add convergence factors}}}{\rightsquigarrow} \prod_{v \nmid \infty} L_v(E, q_v^{-1})^{-1} \int_{E(K_v)} |\omega|_v \cdot \prod_{v \mid \infty} \int_{E(K_v)} |\omega|_v = C_E.$$

It's worth noting that when $v \nmid 2\Delta_E \infty$ (i.e. E/K_v has good reduction), Lemma 1.6 reduces to

$$\left(\frac{\#\tilde{E}(k_v)}{q_v} \right)^{-1} \int_{E(K_v)} |\omega^0|_v = \left| \frac{\omega}{\omega_v^0} \right|_v$$

where the factors $\#\tilde{E}(k_v)/q_v$ were the quantities initially of interest to Birch and Swinnerton-Dyer.

2. TAMAGAWA NUMBERS

Notation 2.1. • \mathcal{K} a non-Archimedean local field, e.g. \mathbb{Q}_p

- $\mathcal{O}_{\mathcal{K}}$ ring of integers, e.g. \mathbb{Z}_p
- k residue field, e.g. \mathbb{F}_p , of cardinality q
- A/\mathcal{K} an abelian variety; C/\mathcal{K} a smooth, proper, geometrically connected curve

The Birch and Swinnerton-Dyer conjecture can be stated more generally for abelian varieties (due to Tate). The product of local terms has a natural generalisation, and at non-Archimedean places the terms are computed from the ‘Néron model’ for A (see [4, Chapter IV, §5] for a definition) – the correct analogue of a minimal regular model for an elliptic curve.

Definition 2.2. A *Néron model* of A is a smooth, separated, finite type group scheme $\mathcal{A}/\mathcal{O}_{\mathcal{K}}$ with generic fibre A , satisfying (the Néron mapping property):

if \mathcal{Y} is a smooth, separated $\mathcal{O}_{\mathcal{K}}$ -scheme, then any \mathcal{K} -morphism $\mathcal{Y}_{\mathcal{K}} \rightarrow A$ can be extended uniquely to an $\mathcal{O}_{\mathcal{K}}$ -morphism $\mathcal{Y} \rightarrow \mathcal{A}$.

Definition 2.3. Let $\mathcal{A}/\mathcal{O}_{\mathcal{K}}$ denote a Néron model for A . Let $\Phi_A := \mathcal{A}_s/\mathcal{A}_s^0$ (the Néron component group/the component group scheme of the special fibre of \mathcal{A}). Then

$$c_{A/\mathcal{K}} := \#\Phi_A(k) = \#\Phi_A(\bar{k})^{\text{Gal}(\bar{k}/k)}.$$

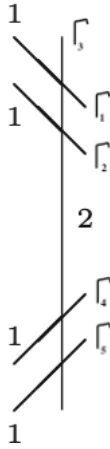
Remark 2.4. (1). Néron models always exist.

(2). The Néron model of E/\mathcal{K} is the smooth part of its minimal regular model and $\Phi_E(k) \cong E(\mathcal{K})/E_0(\mathcal{K})$ (Néron, see [4]). We therefore recover the previous definition of the Tamagawa number, i.e. $c_{E/\mathcal{K}} = [E(\mathcal{K}) : E_0(\mathcal{K})]$.

For higher dimensional A , it is infeasible to compute defining equations in projective space and so the computation of a Néron model is out of scope.

Example 2.8. (Elliptic curve type I_n^*)

Let $n = 0$, i.e. *potential good reduction: acquires good reduction over a field extension.*



Here $I = \{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5\}$. $\forall i \neq 3$ we have $m_i = 1$ and $m_3 = 2$. Clearly,

$$\ker(\beta) = \langle \Gamma_1 - \Gamma_2, \Gamma_2 - \Gamma_4, \Gamma_4 - \Gamma_5, \Gamma_3 - 2\Gamma_1 \rangle_{\mathbb{Z}} = \langle \lambda_1, \lambda_2, \lambda_1 + \lambda_3, \lambda_4 \rangle_{\mathbb{Z}}.$$

The intersection pairing matrix is

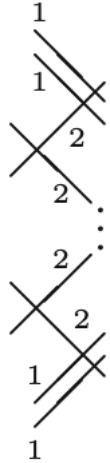
$$M = \begin{bmatrix} -2 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 1 & 1 & -2 & 1 & 1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & -2 \end{bmatrix}$$

which has 4 linearly independent columns. We can show that $\text{im}(\alpha)$ is

$$\langle \lambda_4, 2\lambda_1 + \lambda_4, -3\lambda_1 - 2\lambda_2 - \lambda_3 - 2\lambda_4, 2\lambda_1 + 2\lambda_2 + \lambda_4 \rangle_{\mathbb{Z}} = \langle \lambda_4, 2\lambda_1, \lambda_1 + \lambda_3, 2\lambda_2 \rangle_{\mathbb{Z}}.$$

Therefore $\Phi(\bar{k}) \cong (\mathbb{Z}/2\mathbb{Z})^2$.

Let $n \geq 1$, i.e. *potential multiplicative reduction: acquires multiplicative reduction over a field extension.*



A similar computation gives that

$$\Phi(\bar{k}) \cong \begin{cases} \mathbb{Z}/4\mathbb{Z} & n \text{ odd,} \\ (\mathbb{Z}/2\mathbb{Z})^2 & n \text{ even.} \end{cases}$$

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