MODELS OF CURVES: THE BIRCH AND SWINNERTON-DYER FORMULA

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1. The statement

Notation 1.1. • K a number field of discriminant Δ_K

- E/K an elliptic curve
- $|\cdot|_v$ the normalised absolute value on K_v (for v a place of K)
- q_v the cardinality of the residue field of K_v

Conjecture 1 (Birch and Swinnerton-Dyer part II). Assuming III_E is finite and that L(E, s) has an analytic continuation to \mathbb{C} , its lead coefficient at s = 1 is

$$\frac{\# \amalg_E \cdot \operatorname{Reg}_E \cdot C_E}{\# E(K)_{\operatorname{tors}}^2 \cdot \sqrt{|\Delta_K|}}$$

Todays focus is the product of local terms C_E , which is model dependent.

Definition 1.2. Fix an non-zero differential ω on E. Then

$$C_E := \prod_{v \nmid \infty} c_{E/K_v} \Big| \frac{\omega}{\omega_v^0} \Big|_v \cdot \prod_{\substack{v \mid \infty \\ K_v \cong \mathbb{R}}} \int_{E(K_v)} |\omega|_v \cdot \prod_{\substack{v \mid \infty \\ K_v \cong \mathbb{C}}} \int_{E(K_v)} |\omega \wedge \bar{\omega}| = \prod_v C_{E/K_v}(\omega)$$

where $c_{E/K_v} = [E(K_v) : E_0(K_v)]$ is a Tamagawa number and ω_v^0 is a Néron differential.

If $E: y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ is a minimal Weierstrass equation over K_v , then

$$\omega_v^0 = \pm \frac{dx}{2y + a_1 x + a_3}.$$

Example 1.3. Let E/\mathbb{Q} : $y^2 = x^3 + 17^{12}$, $\Delta_E = -2^3 \cdot 3^3 \cdot 17^{24}$. This is minimal over \mathbb{Q}_p , and so $\omega_p^0 = \pm \frac{dx}{2y}$ whenever $p \neq 17$.

Let $y = 17^{6}Y$, $x = 17^{4}X$. Then $E: Y^{2} = X^{3} + 1$ is minimal over \mathbb{Q}_{17} and

$$\omega_{17}^0 = \pm \frac{dX}{2Y} = \pm 17^2 \frac{dx}{2y}.$$

Remark 1.4. When $K = \mathbb{Q}$, E has a global minimal Weierstrass equation (e.g. $E : y^2 = x^3 + 1$ in the example above). Therefore, there's a canonical choice of ω which gives

$$C_E = \prod_p c_{E/\mathbb{Q}_p} \cdot \int_{E(\mathbb{R})} |\omega|.$$

Remark 1.5. Each local term $C_{E/K_v}(\omega)$ depends on the choice of ω , but C_E doesn't. To see this: for $\alpha \in K^{\times}$, $C_{E/K_v}(\alpha \omega) = |\alpha|_v C_{E/K_v}(\omega)$ and $\prod_v |\alpha|_v = 1$.

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Where does this term come from? (From my understanding).

Recall that $L(E,s) = \prod_{v \nmid \infty} L(E, q_v^{-s})^{-1}$. The non-Archimedean terms appearing in the Birch and Swinnerton-Dyer conjecture arise naturally as v-adic integrals.

Lemma 1.6 (Tate, [5] Theorem 5.2). Given $v \nmid \infty$,

$$L_{v}(E, q_{v}^{-1})^{-1} \int_{E(K_{v})} |\omega^{0}|_{v} = c_{E/K_{v}} \left| \frac{\omega}{\omega_{v}^{0}} \right|_{v}.$$

In the 1950's, before the formulation of the Birch and Swinnerton-Dyer conjecture, the Tamagawa number of a semisimple algebraic group was receiving a lot of attention. It is possible that Birch and Swinnerton-Dyer tried to mimic this construction for elliptic curves as follows:

$$\prod_{v} \int_{E(K_{v})} |\omega|_{v} \xrightarrow{\text{doesn't convergen}}_{\text{add convergence factors}} \prod_{v \nmid \infty} L_{v}(E, q_{v}^{-1})^{-1} \int_{E(K_{v})} |\omega|_{v} \cdot \prod_{v \mid \infty} \int_{E(K_{v})} |\omega|_{v} = C_{E}$$

It's worth noting that when $v \nmid 2\Delta_E \infty$ (i.e. E/K_v has good reduction), Lemma 1.6 reduces to

$$\left(\frac{\#\tilde{E}(k_v)}{q_v}\right)^{-1}\int_{E(K_v)}|\omega^0|_v = \left|\frac{\omega}{\omega_v^0}\right|_v$$

where the factors $\#\tilde{E}(k_v)/q_v$ were the quantities initially of interest to Birch and Swinnerton-Dyer.

2. TAMAGAWA NUMBERS

Notation 2.1. • \mathcal{K} a non-Archimedean local field, e.g. \mathbb{Q}_p

- $\mathcal{O}_{\mathcal{K}}$ ring of integers, e.g. \mathbb{Z}_p
- k residue field, e.g. \mathbb{F}_p , of cardinality q
- A/\mathcal{K} an abelian variety; C/\mathcal{K} a smooth, proper, geometrically connected curve

The Birch and Swinnerton-Dyer conjecture can be stated more generally for abelian varieties (due to Tate). The product of local terms has a natural generalisation, and at non-Archimedean places the terms are computed from the 'Néron model' for A (see [4, Chapter IV, §5] for a definition) – the correct analogue of a minimal regular model for an elliptic curve.

Definition 2.2. A *Néron model* of A is a smooth, separated, finite type group scheme $\mathcal{A}/\mathcal{O}_{\mathcal{K}}$ with generic fibre A, satisfying (the Néron mapping property):

if \mathcal{Y} is a smooth, separated $\mathcal{O}_{\mathcal{K}}$ -scheme, then any \mathcal{K} -morphism $\mathcal{Y}_{\mathcal{K}} \to A$ can be extended uniquely to an $\mathcal{O}_{\mathcal{K}}$ -morphism $\mathcal{Y} \to \mathcal{A}$.

Definition 2.3. Let $\mathcal{A}/\mathcal{O}_{\mathcal{K}}$ denote a Néron model for A. Let $\Phi_A := \mathcal{A}_s/\mathcal{A}_s^0$ (the Néron component group/the component group scheme of the special fibre of \mathcal{A}). Then

$$c_{A/\mathcal{K}} := \#\Phi_A(k) = \#\Phi_A(\bar{k})^{\operatorname{Gal}(k/k)}.$$

Remark 2.4. (1). Néron models always exist.

(2). The Néron model of E/\mathcal{K} is the smooth part of its minimal regular model and $\Phi_E(k) \cong E(\mathcal{K})/E_0(\mathcal{K})$ (Néron, see [4]). We therefore recover the previous definition of the Tamagawa number, i.e. $c_{E/\mathcal{K}} = [E(\mathcal{K}) : E_0(\mathcal{K})]$.

For higher dimensional A, it is infeasible to compute defining equations in projective space and so the computation of a Néron model is out of scope.

If $A = \operatorname{Jac} C$, it turns out that the Néron model can be described in terms of the minimal regular model of C (a theorem of Raynaud, [3, Theorem 9.5.4]). This allows us to compute the Tamagawa number for such an A as follows.

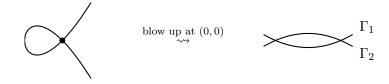
Theorem 2.5 ([2] Theorem 1.1). Let $\mathcal{C}/\mathcal{O}_{\mathcal{K}}$ be a minimal regular model for C. Write I = $\{\Gamma_1,\ldots,\Gamma_n\}$ for the irreducible components of \mathcal{C}_s base-changed to k and $m_i \in \mathbb{N}$ for the multiplicity of Γ_i . Define (extending linearly)

$$\alpha: \ \mathbb{Z}^I \longrightarrow \mathbb{Z}^I \qquad \qquad \beta: \ \mathbb{Z}^I \longrightarrow \mathbb{Z}$$
$$\Gamma_i \longmapsto \sum_{1 \le j \le n} (\Gamma_i \cdot \Gamma_j) \Gamma_j \qquad \qquad \Gamma_i \longmapsto m_i.$$

Then $\Phi_{\operatorname{Jac} C}(\overline{k}) \cong \operatorname{ker}(\beta)/\operatorname{im}(\alpha)$. In particular,

$$c_{\operatorname{Jac} C/\mathcal{K}} = \#\operatorname{ker}(\beta)/\operatorname{im}(\alpha)^{\operatorname{Gal}(k/k)}$$

Example 2.6. (Elliptic curve type I_2) Let $E: y^2 = x^3 + 2x^2 + p^2$ over \mathbb{Q}_p for p odd. This is a minimal Weierstrass equation for E whose reduction is a nodal cubic curve. Blowing up once here yields the minimal regular model:



Let C have such a minimal regular model. Write $I = \{\Gamma_1, \Gamma_2\}, m_1 = m_2 = 1$ and $(\Gamma_1 \cdot \Gamma_2) =$ $(\Gamma_2 \cdot \Gamma_1) = 2$ and $(\Gamma_1 \cdot \Gamma_1) = (\Gamma_1 \cdot \Gamma_2) = -2$. Then $\ker(\beta) = \langle \Gamma_2 - \Gamma_1 \rangle_{\mathbb{Z}}$ and $\operatorname{im}(\alpha) = \langle 2(\Gamma_2 - \Gamma_1) \rangle_{\mathbb{Z}}$ (since $\alpha(\Gamma_1) = 2(\Gamma_2 - \Gamma_1) = -\alpha(\Gamma_2)$. Therefore,

$$\Phi_{\operatorname{Jac} C}(\bar{k}) \cong \mathbb{Z}/2\mathbb{Z} = \{0, \Gamma_2 - \Gamma_1\}.$$

If Gal(k/k) acts trivially (for E this corresponds to (2/p) = +1), then c_{Jac C/K} = 2.
If Gal(k/k) acts non-trivially, (for E this corresponds to (2/p) = −1), then c_{Jac C/K} = 2 too.

Example 2.7. Let $n \geq 3$. Let C have the given minimal regular model (e.g. an elliptic curve of type I_n).

M has n-1 linearly independent columns and we can show that $im(\alpha)$ is

$$M \text{ has } n = 1 \text{ interfy independent columns and we can show that } im(\alpha) \text{ is } \\ \langle \Gamma_1 - 2\Gamma_2 + \Gamma_3, \dots, \Gamma_{n-2} - 2\Gamma_{n-1} + \Gamma_n, n(\Gamma_{n-1} - \Gamma_n) \rangle_{\mathbb{Z}}.$$

$$Therefore \Phi_{\text{Jac } C}(\bar{k}) \cong \mathbb{Z}/n\mathbb{Z}.$$

$$\bullet \text{ If } \text{Gal}(\bar{k}/k) \text{ acts trivially, then } c_{\text{Jac } C/\mathcal{K}} = n.$$

$$\bullet \text{ If } \text{Gal}(\bar{k}/k) \text{ acts non-trivially, then when } n \text{ is } \text{odd} \Rightarrow c_{\text{Jac } C/\mathcal{K}} = 1 \text{ and when } n \text{ is even } \Rightarrow c_{\text{Jac } C/\mathcal{K}} = 2.$$

$$M = \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 1 & \ddots & \ddots & 0 \\ 0 & 1 & -2 & \ddots & \ddots & 1 & 0 \\ 0 & \ddots & \ddots & 1 & 0 \\ 0 & \ddots & \ddots & 1 & -2 & 1 \\ 1 & 0 & \cdots & 0 & 1 & -2 \end{bmatrix}.$$

If $\operatorname{Gal}(\bar{k}/k)$ acts non-trivially, then when n is odd $\Rightarrow c_{\operatorname{Jac} C/\mathcal{K}} = 1$ and when n is even \Rightarrow $c_{\operatorname{Jac} C/\mathcal{K}} = 2.$ $c_{\operatorname{Jac} C/\mathcal{K}} = 2.$

Example 2.8. (Elliptic curve type I_n^*)

Let n = 0, i.e. potential good reduction: acquires good reduction over a field extension.

Here
$$I = \{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5\}$$
. $\forall i \neq 3$ we have $m_i = 1$ and $m_3 = 2$. Clearly,

$$\ker(\beta) = \langle \Gamma_1 - \Gamma_2, \Gamma_2 - \Gamma_4, \Gamma_4 - \Gamma_5, \Gamma_3 - 2\Gamma_1 \rangle_{\mathbb{Z}} = \langle \lambda_1, \lambda_2, \lambda_1 + \lambda_3, \lambda_4 \rangle_{\mathbb{Z}}.$$
The intersection pairing matrix is

$$M = \begin{bmatrix} -2 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 1 & 1 & -2 & 1 & 1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & -2 \end{bmatrix}$$
which has 4 linearly independent columns. We can show that $\operatorname{im}(\alpha)$ is

$$\langle \lambda_4, 2\lambda_1 + \lambda_4, -3\lambda_1 - 2\lambda_2 - \lambda_3 - 2\lambda_4, 2\lambda_1 + 2\lambda_2 + \lambda_4 \rangle_{\mathbb{Z}} = \langle \lambda_4, 2\lambda_1, \lambda_1 + \lambda_3, 2\lambda_2 \rangle_{\mathbb{Z}}.$$
Therefore $\Phi(\bar{k}) \cong (\mathbb{Z}/2\mathbb{Z})^2.$

Let $n \ge 1$, i.e. potential multiplicative reduction: acquires multiplicative reduction over a field extension.

A similar computation gives that

$$\Phi(\bar{k}) \cong \begin{cases} \mathbb{Z}/4\mathbb{Z} & n \text{ odd,} \\ (\mathbb{Z}/2\mathbb{Z})^2 & n \text{ even.} \end{cases}$$

References

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