An arithmetic analogue of the parity conjecture

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Main results

Theorem (Dokchitser, Green, Konstantinou, Morgan)

Assuming $\#\text{Ш}$ is finite, for all smooth, projective curves over number fields X/K

$$\operatorname{rank}(\mathrm{Jac}_X) \equiv \sum_{v \text{ place of } K} \Lambda(X/K_v) \mod 2$$

where $\Lambda \in \{0, 1\}$ is an explicit invariant computed from curves over local fields.

Theorem (Green)

Assuming $\#\text{Ш}$ is finite, the Birch and Swinnerton-Dyer conjecture correctly predicts the parity of the rank of elliptic curves over number fields.

Ranks of elliptic curves

Let $E/\mathbb{Q}$ be an elliptic curve.

Theorem (Mordell)

$E(\mathbb{Q}) \cong \mathbb{Z}^{\mathrm{rank}(E)} \times T$ for some $\mathrm{rank}(E) \in \mathbb{N}$ and finite group T.

Conjecture (Birch and Swinnerton-Dyer I)

$$\mathrm{rank}(E) = \mathrm{ord}_{s=1} L(E,s).$$

Functional equation

$$L^*(E,s) = w(E) L^*(E, 2-s), \qquad w(E) \in \{\pm 1\}.$$

$$(-1)^{\mathrm{ord}_{s=1} L(E,s)} = w(E) := \prod_{v \text{ place of } \mathbb{Q}} w_v(E).$$

The parity conjecture

$$(-1)^{\text{rank}(E)} = w(E) := \prod_{v \text{ place of } \mathbb{Q}} w_v(E)$$

$$w_\infty(E) = -1, \qquad w_p(E) = \begin{cases} +1 & E/\mathbb{Q}_p \text{ has good reduction} \\ -1 & E/\mathbb{Q}_p \text{ has split multiplicative reduction} \\ +1 & E/\mathbb{Q}_p \text{ has non-split multiplicative reduction} \\ \ldots & E/\mathbb{Q}_p \text{ has additive reduction} \end{cases}$$

Let $E/\mathbb{Q} : y^2 = x^3 + 4x^2 - 80x + 400$, $\Delta_E = -5^3 \cdot 11 \cdot 13$. Then

$$w(E) = w_\infty(E) w_5(E) w_{11}(E) w_{13}(E) = (-1)(-1)(+1)(-1) = -1.$$

The parity conjecture says that E has **odd** rank $\Rightarrow$ E has infinitely many rational points.

Parity phenomena

For semistable elliptic curves over number fields,

$$(-1)^{\mathrm{rank}(E)} = (-1)^{\#\{v|\infty\} + \#\{v\nmid\infty,\ E/K_v \text{ split multiplicative}\}}.$$

If $E/\mathbb{Q}$ is semistable with split multiplicative reduction at 2 then $\mathrm{rank}(E/\mathbb{Q}(\zeta_8))$ is odd.

If K is imaginary quadratic and E/K has everywhere good reduction then $\mathrm{rank}(E/K)$ is odd.
If L/K has even degree then $\mathrm{rank}(E/L)$ is even and

$$\mathrm{rank}(E/K) < \mathrm{rank}(E/L).$$

Goal

Develop an arithmetic analogue of the parity conjecture,

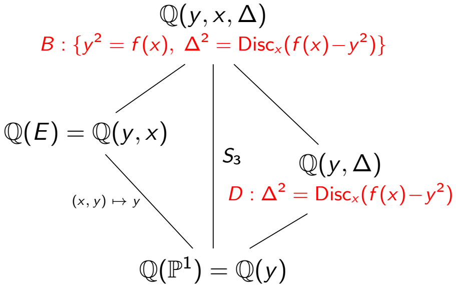
$$(-1)^{\mathrm{rank}(E)} = \prod_{v \text{ place of } K} (-1)^{\Lambda_v(E)} \quad \text{or} \quad \mathrm{rank}(E) \equiv \sum_{v \text{ place of } K} \Lambda_v(E) \mod 2.$$

New idea: use the arithmetic of higher genus curves.

Taking covers of curves

Let $E/\mathbb{Q} : y^2 = f(x)$ be an elliptic curve. If $f(x) = x^3 + ax + b$

$$\implies D : \Delta^2 = -27y^4 + 54by^2 - (4a^3 + 27b^2).$$



Example: B has genus 3

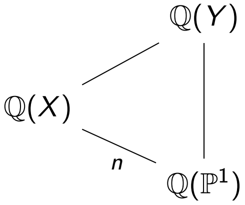
$\Omega^1(B) = \mathbb{1}^{\oplus a} \oplus \epsilon^{\oplus b} \oplus \rho^{\oplus c} \Rightarrow B$ has genus $a + b + 2c$.

- $0 = \dim \Omega^1(\mathbb{P}^1) = \dim \Omega^1(B)^{S_3} = a$,
- $1 = \dim \Omega^1(D) = \dim \Omega^1(B)^{C_3} = b$,
- $1 = \dim \Omega^1(E) = \dim \Omega^1(B)^{C_2} = c$.

Theorem

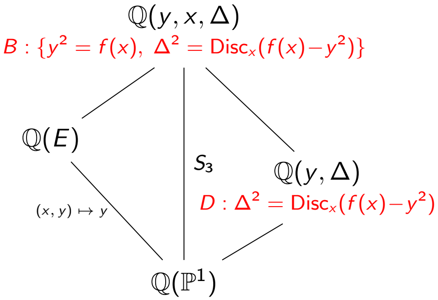
Let $Y/\mathbb{Q}$ be a smooth, projective curve and $G \leq \mathrm{Aut}_{\mathbb{Q}}(Y)$.

- $\mathbb{Q}(Y)^G = \mathbb{Q}(Y/G)$,
- $\Omega^1(Y)^G = \Omega^1(Y/G)$,
- $(\mathrm{Jac}_Y(\mathbb{Q}) \otimes \mathbb{Q})^G = \mathrm{Jac}_{Y/G}(\mathbb{Q}) \otimes \mathbb{Q}$.



Finding a relationship between E, D, B, $\mathbb{P}^1$

Let $E/\mathbb{Q} : y^2 = f(x)$ be an elliptic curve.



E	rank(E)	rank(Jac_D)	rank(Jac_B)
$y^2 = x^3 + x + 1$	1	1	3
$y^2 = x^3 - 3x + 1$	1	0	2
$y^2 = x^3 - 16x + 400$	3	1	7

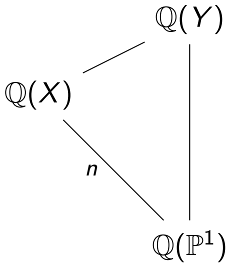
$$2\mathrm{rank}(E) + \mathrm{rank}(\mathrm{Jac}_D) = \mathrm{rank}(\mathrm{Jac}_B)$$

Theorem

There's an isogeny

$$E \times E \times \mathrm{Jac}_D \to \mathrm{Jac}_B.$$

Let $X/\mathbb{Q}$ be a smooth, projective curve and $\pi : X \to \mathbb{P}^1$.



$\sum_i H_i - \sum_j H_j'$ is *a Brauer relation for a finite group G* if

$$\sum_i \mathrm{Ind}_{H_i}^G \mathbb{1} = \sum_j \mathrm{Ind}_{H_j'}^G \mathbb{1}.$$

A Brauer relation for S_3 is

$$C_2 + C_2 + C_3 - \{1\} - S_3 - S_3.$$

Theorem (Kani–Rosen)

Let $Y/\mathbb{Q}$ be a smooth, projective curve and $G \leq \mathrm{Aut}_{\mathbb{Q}}(Y)$. If $\sum_i H_i - \sum_j H_j'$ is a Brauer relation for G, then there's an isogeny

$$\prod_i \mathrm{Jac}_{Y/H_i} \longrightarrow \prod_j \mathrm{Jac}_{Y/H_j'}.$$

Isogeny invariance of BSD

$$\mathrm{BSD}_{\mathrm{Jac}_X} := \frac{\#\text{Ш}_{\mathrm{Jac}_X} \cdot \mathrm{Reg}_{\mathrm{Jac}_X} \cdot C_{\mathrm{Jac}_X}}{\#\mathrm{Jac}_X(\mathbb{Q})^2_{\mathrm{tors}}}$$

Theorem (Cassels–Tate)

Assume that $\#\text{Ш}$ is finite. The BSD coefficient is invariant under isogeny.

Apply to the isogeny $E \times E \times \mathrm{Jac}_D \to \mathrm{Jac}_B$.

$$\square \cdot 3^{\mathrm{rank}(E)+\mathrm{rank}(\mathrm{Jac}_D)} = \frac{\mathrm{Reg}_{\mathrm{Jac}_B}}{\mathrm{Reg}_E^2\ \mathrm{Reg}_{\mathrm{Jac}_D}} = \frac{\#\mathrm{Jac}_B(\mathbb{Q})^2_{\mathrm{tors}}}{\#E(\mathbb{Q})^4_{\mathrm{tors}}\ \#\mathrm{Jac}_D(\mathbb{Q})^2_{\mathrm{tors}}} \cdot \frac{\#\text{Ш}_E^2\ \#\text{Ш}_{\mathrm{Jac}_D}}{\#\text{Ш}_{\mathrm{Jac}_B}} \cdot \frac{C_E^2\ C_{\mathrm{Jac}_D}}{C_{\mathrm{Jac}_B}} = \square \cdot \frac{C_E^2\ C_{\mathrm{Jac}_D}}{C_{\mathrm{Jac}_B}}$$

Theorem

Assuming that $\#\text{Ш}_E[3^\infty]$ and $\#\text{Ш}_{\mathrm{Jac}_D}[3^\infty]$ are finite,

$$\mathrm{rank}(E) + \mathrm{rank}(\mathrm{Jac}_D) \equiv \mathrm{ord}_3\left(\frac{c_\infty(E)^2 c_\infty(\mathrm{Jac}_D)}{c_\infty(\mathrm{Jac}_B)}\right) + \sum_p \mathrm{ord}_3\left(\frac{c_p(E)^2 c_p(\mathrm{Jac}_D)}{c_p(\mathrm{Jac}_B)}\right) \mod 2.$$

Example

$$\operatorname{rank}(E) + \operatorname{rank}(\mathrm{Jac}_D) \equiv \sum_{v=p,\infty} \operatorname{ord}_3 \left(\frac{c_v(E)^2 c_v(\mathrm{Jac}_D)}{c_v(\mathrm{Jac}_B)} \right) \mod 2$$

$$E/\mathbb{Q} : y^2 = x^3 + x^2 - 9x - \frac{59}{4} \quad (19.\mathrm{a}2), \qquad D/\mathbb{Q} : \Delta^2 = \mathrm{Disc}_x\left(x^3 + x^2 - 9x - \frac{59}{4} - y^2\right)$$

$$= -27y^4 - \frac{1261}{2}y^2 - \frac{6859}{16}.$$

Jac_D is 798.d4.

v	$c_v(E)$	$c_v(\mathrm{Jac}_D)$	$c_v(\mathrm{Jac}_B)$	$\operatorname{ord}_3 \left(\frac{c_v(E)^2 c_v(\mathrm{Jac}_D)}{c_v(\mathrm{Jac}_B)} \right)$
2	1	2	2	0
3	1	3	1	1
7	1	6	2	1
19	3	3	27	0
∞	$1.3598\ldots$	$0.5121\ldots$	$0.9469\ldots$	0

$\implies \operatorname{rank}(E) + \operatorname{rank}(\mathrm{Jac}_D)$ is even.

Obtaining local formulae

Theorem (Dokchitser, Green, Konstantinou, Morgan)

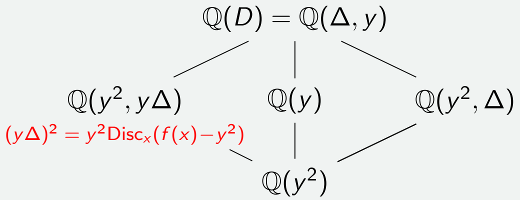
Let $Y/\mathbb{Q}$ be smooth, projective such that $\#Ш_{\mathrm{Jac}_Y}[\ell^\infty]$ is finite. Assume $Y \to \mathbb{P}^1$ is a Galois cover and let $\Theta = \sum_i H_i - \sum_j H'_j$ be a Brauer relation for its Galois group. Then

$$\mathrm{ord}_\ell \left(\frac{\prod_i \mathrm{Reg}_{\mathrm{Jac}_{Y/H_i}}}{\prod_j \mathrm{Reg}_{\mathrm{Jac}_{Y/H'_j}}} \right) \equiv \sum_{v=p,\infty} \Lambda_{v,\Theta}(Y) \mod 2.$$

$D : \Delta^2 = \mathrm{Disc}_x(f(x) - y^2)$ is acted on by $C_2 \times C_2$

$$\Rightarrow \mathrm{rank}(\mathrm{Jac}_D) \equiv \sum_{v=p,\infty} \Lambda_{v,\Theta}(D) \mod 2$$

where $\Theta = C_2^a + C_2^b + C_2^c - 2C_2 \times C_2 - \{1\}$.



$$\Longrightarrow \mathrm{rank}(E) + \mathrm{rank}(\mathrm{Jac}_D) + \mathrm{rank}(\mathrm{Jac}_D) \equiv \sum_{v=p,\infty} \Lambda_{v,\Theta'}(B) + \Lambda_{v,\Theta}(D) \mod 2.$$

An arithmetic analogue of the parity conjecture

Theorem (Dokchitser, Green, Konstantinou, Morgan)

Assume $\#\text{Ш}$ *is finite. Let* $X/\mathbb{Q}$ *be a smooth, projective curve. There is a finite collection of Brauer relations* Br *such that*

$$\mathrm{rank}(\mathrm{Jac}_X) \equiv \sum_{v=p,\infty} \sum_{\Theta \in \mathrm{Br}} \Lambda_{v,\Theta} \mod 2.$$

Equivalently, there's an explicit invariant $\Lambda_v \in \mathbb{Z}$ *computed from curves over local fields such that*

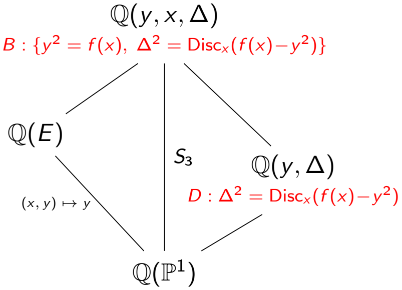
$$(-1)^{\mathrm{rank}(\mathrm{Jac}_X)} = \prod_{v=p,\infty} (-1)^{\Lambda_v}.$$

The parity conjecture

$$(-1)^{\mathrm{rank}(\mathrm{Jac}_X)} = \prod_{v=p,\infty} w_v(\mathrm{Jac}_X).$$

Summary

Let $E/\mathbb{Q} : y^2 = f(x)$ be an elliptic curve.



$$E \times E \times \mathrm{Jac}_D \to \mathrm{Jac}_B$$

Assume that $\#Ш_E[3^\infty]$ and $\#Ш_{\mathrm{Jac}_D}[3^\infty]$ are finite,

$$\mathrm{rank}(E) + \mathrm{rank}(\mathrm{Jac}_D) \equiv \sum_{v=p,\infty} \Lambda_v(B) \mod 2.$$

Assume $\#Ш_E[3^\infty]$, $\#Ш_{\mathrm{Jac}_D}[3^\infty]$, $\#Ш_{\mathrm{Jac}_D}[2^\infty]$ are finite,

$$\mathrm{rank}(E) \equiv \sum_{v=p,\infty} \Lambda_v(B) + \Lambda_v(D) \mod 2.$$

Theorem

Assume $\#Ш$ is finite. Let $X/\mathbb{Q}$ be a smooth, projective curve. Then,

$$(-1)^{\mathrm{rank}(\mathrm{Jac}_X)} = \prod_{v=p,\infty} (-1)^{\Lambda_v}.$$

Example

$$E/\mathbb{Q} : y^2 = x^3 + x^2 - 9x - \frac{59}{4} \quad (19.a2), \qquad D/\mathbb{Q} : \Delta^2 = -27y^4 - \frac{1261}{2}y^2 - \frac{6859}{16}.$$

$$(-1)^{\mathrm{rank}(E)+\mathrm{rank}(\mathrm{Jac}_D)} = \prod_{v=p,\infty} w_v(E) w_v(\mathrm{Jac}_D)$$

v	$c_v(E)$	$c_v(\mathrm{Jac}_D)$	$c_v(\mathrm{Jac}_B)$	$\mathrm{ord}_3\left(\frac{c_v(E)^2 c_v(\mathrm{Jac}_D)}{c_v(\mathrm{Jac}_B)}\right)$	$w_v(E)$	$w_v(\mathrm{Jac}_D)$
2	1	2	2	0	1	1
3	1	3	1	1	1	-1
7	1	6	2	1	1	-1
19	3	3	27	0	-1	-1
∞	$1.3598\ldots$	$0.5121\ldots$	$0.9469\ldots$	0	-1	-1

Theorem (Green)

$$(-1)^{\mathrm{ord}_3\left(\frac{c_v(E)^2 c_v(\mathrm{Jac}_D)}{c_v(\mathrm{Jac}_B)}\right)} = w_v(E) w_v(\mathrm{Jac}_D) \qquad \text{when } v = p, \infty.$$

Proving the parity conjecture for E

Theorem (Green)

Let E/K be an elliptic curve. Assume that $\#\text{Ш}_{E/K}[3^\infty]$, $\#\text{Ш}_{\text{Jac}_D/K}[3^\infty]$, $\#\text{Ш}_{\text{Jac}_D/K}[2^\infty]$ are finite. The parity conjecture holds for E.

Proof.

Assume that $\#\text{Ш}_{E/K}[3^\infty]$, $\#\text{Ш}_{\text{Jac}_D/K}[3^\infty]$ are finite. By the previous theorems,

$$(-1)^{\text{rank}(E)+\text{rank}(\text{Jac}_D)} = \prod_v (-1)^{\text{ord}_3\left(\frac{c_v(E)^2 c_v(\text{Jac}_D)}{c_v(\text{Jac}_B)}\right)} = \prod_v w_v(E) w_v(\text{Jac}_D) = w(E)w(\text{Jac}_D).$$

Assume that $\#\text{Ш}_{\text{Jac}_D/K}[2^\infty]$ is finite. Dokchitser–Dokchitser have shown that

$$(-1)^{\text{rank}(\text{Jac}_D)} = w(\text{Jac}_D).$$

□

Further applications to the parity conjecture

Theorem (Green)

Assume $\#\text{Ш}$ is finite. The parity conjecture holds for elliptic curves over number fields.

Theorem (Green, Maistret)

The p-parity conjecture holds for elliptic curves over totally real fields.

Work in progress (Dokchitser, Green, Morgan)

Assume $\#\text{Ш}$ is finite. The parity conjecture holds for Jacobians of semistable hyperelliptic curves over number fields with good ordinary reduction at places $v \mid 2$.

Thank you for your attention!